Anisotropic cosmography of the local Universe

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Outline

Introduction to the field of cosmography

Motivation for anisotropic cosmography approaches

Anisotropic cosmography for cosmic distances

Anisotropic cosmography for redshift drift and position drift

Discussion   ⋆ things to think about in applying cosmography to data

Cosmography

Investigation of the Universe through cosmological measurements without imposing field equations (S. Weinberg, M. Visser)

Mapping of cosmological kinematics (as opposed to dynamics)

Extracting information from data that is valid irrespective of matter content or theory of gravity

Usual approaches to cosmography assume homogeneity and isotropy, i.e., Friedmann-Lemaître-Robertson-Walker metrics

Anisotropic cosmography

Some approaches to cosmography go further and stay agnostic to the field equations *and* the space-time metric

Allows us to map the kinematics/curvature in our cosmic vicinity where the space-time has not converged to being isotropic

This approach to cosmography is ambitious (introduces extra degrees of freedom) but also extremely rewarding if successful

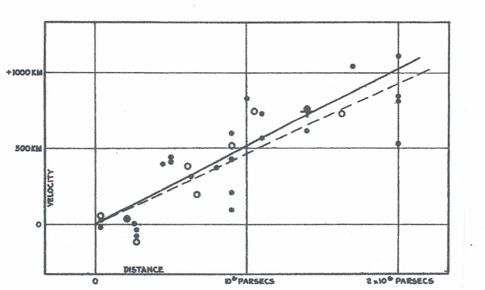
Complementary to isotropic analysis and more model-dependent analysis

Provide a route for making rigorous investigations of anisotropies in cosmological parameters.

Cosmography in practice

Relies on series expansions of observables around the observer (typically with redshift as a parameter).

Example of cosmography: Measurement of the expansion of space through the Hubble-Lemaître law $d_L \approx z/H_0$: G. Lemaître (1927), V. M. Slipher (1917), E. P. Hubble (1929).



E. P. Hubble (1929): Velocity-Distance Relation among Extra-Galactic Nebulae. ©US National Academy of Sciences.

Luminosity distance

Distance defined by the ratio of intrinsic luminosity and flux observed from an astronomical object

⋆ Standard candles: Supernovae of type Ia
⋆ Standard sirens: Gravitational waves from black hole and neutron star mergers

Supernova remnant.
credit: NASA, ESA, and J. Banovetz, D. Milisavljevic

Cosmography, **FLRW geometry**

The luminosity distance cosmography, M. Visser (2004):

$$d_L = \frac{1}{H_0} z + \frac{1-q_0}{2H_0} z^2 + \frac{-1+3q_0^2+q_0-j_0+\Omega_{k0}}{6H_0} z^3 + \mathcal{O}(z^4)$$

$$H \equiv \frac{\dot{a}}{a}\,, \qquad q \equiv -\frac{\ddot{a}}{aH^2}\,, \qquad j \equiv \frac{\dddot{a}}{aH^3}\,, \qquad \Omega_k \equiv \frac{-k}{a^2H^2}$$

$$\dot{} \equiv \frac{d}{dt}\,, \qquad k \in \{-1, 0, 1\}$$

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$$\dot{} \equiv \frac{d}{dt}\,, \qquad k \in \{-1, 0, 1\}$$

Purely geometrical result that is useful for analysing data in the isotropic regime

In the presence of structures: Observables are dependent on the direction on the sky, and we must go beyond FLRW cosmography

Luminosity distance cosmography, **general geometry**

Isotropic FLRW cosmography $\rightarrow$ cosmography without symmetries

Important papers J. Kristian and R. K. Sachs (1966), S. Seitz, P. Schneider and J. Ehlers (1994), C. Clarkson and O. Umeh (2011), C. Clarkson, G. F. R. Ellis, A. Faltenbacher, R. Maartens, O. Umeh, J. P. Uzan (2012)

Generalising the luminosity distance cosmography:

$$d_L = \frac{1}{\mathfrak{H}_o} z + \frac{1-\mathfrak{Q}_o}{2\mathfrak{H}_o} z^2 + \frac{-1+3\mathfrak{Q}_o^2+\mathfrak{Q}_o-\mathfrak{J}_o+\mathfrak{R}_o}{6\mathfrak{H}_o} z^3 + \mathcal{O}(z^4)$$

$$H \rightarrow \mathfrak{H}\,, \qquad q \rightarrow \mathfrak{Q}\,, \qquad j \rightarrow \mathfrak{J}\,, \qquad \Omega_k \rightarrow \mathfrak{R}$$

Generalised cosmological parameters $\mathfrak{H}\,, \mathfrak{Q}\,, \mathfrak{J}\,, \mathfrak{R}$ vary with the point of observation and the line of sight.

Luminosity distance cosmography, **general geometry**

A. Heinesen, JCAP05(2021)008 [arXiv:2010.06534]

Assumptions: (i) Lorentzian space-time;
(ii) geometrical optics;
(iii) Etherington's reciprocity theorem;
(iv) congruence description for emitters and observer of light/gravitational waves

The luminosity distance cosmography for a general congruence of observers and emitters in a general space-time:

$$d_L = \frac{1}{\mathfrak{H}_o} z + \frac{1-\mathfrak{Q}_o}{2\mathfrak{H}_o} z^2 + \frac{-1+3\mathfrak{Q}_o^2+\mathfrak{Q}_o-\mathfrak{J}_o+\mathfrak{R}_o}{6\mathfrak{H}_o} z^3 + \mathcal{O}(z^4)$$

$$\mathfrak{H} = -\frac{\frac{\mathrm{d}E}{\mathrm{d}\lambda}}{E^2}\,, \qquad \mathfrak{Q} = -1 - \frac{1}{E}\frac{\frac{\mathrm{d}\mathfrak{H}}{\mathrm{d}\lambda}}{\mathfrak{H}^2}\,,$$

$$\mathfrak{R} = 1 + \mathfrak{Q} - \frac{1}{2E^2}\frac{k^\mu k^\nu R_{\mu\nu}}{\mathfrak{H}^2}\,, \qquad \mathfrak{J} = \frac{1}{E^2}\frac{\frac{\mathrm{d}^2\mathfrak{H}}{\mathrm{d}\lambda^2}}{\mathfrak{H}^3} - 4\mathfrak{Q} - 3$$

Luminosity distance cosmography, **general geometry**

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$\frac{\mathrm{d}E}{\mathrm{d}\lambda}$: rate of change of photon energy, E, along null ray.

$\frac{\mathrm{d}}{\mathrm{d}\lambda} \equiv k^\mu \nabla_\mu$: Derivative along photon 4-momentum, k^μ.

Luminosity distance cosmography, **general geometry**

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$\frac{\mathrm{d}E}{\mathrm{d}\lambda}$: rate of change of photon energy, E, along null ray.

$\frac{\mathrm{d}}{\mathrm{d}\lambda} \equiv k^\mu \nabla_\mu$: Derivative along photon 4-momentum, k^μ.

$k^\mu k^\nu R_{\mu\nu}$: Ricci focusing term.

Interpretation of effective cosmological parameters

⋆ The effective Hubble parameter

Ellis et al. (1984)

Multipole expansion of $\mathfrak{H}$ in direction of incoming light e^μ:

$$\mathfrak{H} = \frac{1}{3}\theta - e^\mu a_\mu + e^\mu e^\nu \sigma_{\mu\nu} \,,$$

$$9 \text{ dof} \begin{cases} \theta \text{ : expansion of observer congruence} \\ a^\mu \text{ : 4–acceleration of observer congruence} \\ \sigma_{\mu\nu} \text{ : shear of observer congruence} \end{cases}$$

Interpretation of effective cosmological parameters

Multipole expansions in direction of incoming light, e^{μ}

Physically interpretable multipole coefficients

$$H \rightarrow \mathfrak{H} = \frac{1}{3}\theta - e^{\mu} a_{\mu} + e^{\mu} e^{\nu} \sigma_{\mu\nu} \,, \qquad 1 \text{ dof} \rightarrow 9 \text{ dof}$$

$$9 \text{ dof} \begin{cases} \theta : \text{expansion of observer congruence} \\ a^{\mu} : \text{4–acceleration of observer congruence} \\ \sigma_{\mu\nu} : \text{shear of observer congruence} \end{cases}$$

$$q \rightarrow \mathfrak{Q} = -1 - \frac{\overset{0}{\mathfrak{q}} + e^{\mu} \overset{1}{\mathfrak{q}}_{\mu} + e^{\mu} e^{\nu} \overset{2}{\mathfrak{q}}_{\mu\nu} + e^{\mu} e^{\nu} e^{\rho} \overset{3}{\mathfrak{q}}_{\mu\nu\rho} + e^{\mu} e^{\nu} e^{\rho} e^{\kappa} \overset{4}{\mathfrak{q}}_{\mu\nu\rho\kappa}}{\mathfrak{H}^{2}(\boldsymbol{e})} \,,$$

1 dof → 16 independent dof

$$j_0 - \Omega_{k0} \rightarrow \mathfrak{J} - \mathfrak{R} = 1 + \frac{\overset{0}{\mathfrak{t}} + \boldsymbol{e} \cdot \overset{1}{\mathfrak{t}} + \boldsymbol{ee} \cdot \overset{2}{\mathfrak{t}} + \boldsymbol{eee} \cdot \overset{3}{\mathfrak{t}} + \boldsymbol{eeee} \cdot \overset{4}{\mathfrak{t}} + \boldsymbol{eeeee} \cdot \overset{5}{\mathfrak{t}} + \boldsymbol{eeeeee} \cdot \overset{6}{\mathfrak{t}}}{\mathfrak{H}^{3}(\boldsymbol{e})} \,,$$

1 dof → 36 independent dof

Interpretation of effective cosmological parameters

⋆ The effective deceleration parameter

Multipole expansion of $\mathfrak{Q}$ in direction of incoming light e^μ:

$$\mathfrak{Q} = -1 - \frac{\overset{0}{\mathfrak{q}} + e^\mu \overset{1}{\mathfrak{q}}_\mu + e^\mu e^\nu \overset{2}{\mathfrak{q}}_{\mu\nu} + e^\mu e^\nu e^\rho \overset{3}{\mathfrak{q}}_{\mu\nu\rho} + e^\mu e^\nu e^\rho e^\kappa \overset{4}{\mathfrak{q}}_{\mu\nu\rho\kappa}}{\mathfrak{H}^2(\boldsymbol{e})}, \quad 16 \text{ idof}$$

$$\overset{0}{\mathfrak{q}} \equiv \frac{1}{3}\frac{\mathrm{d}\theta}{\mathrm{d}\tau} + \frac{1}{3}D_\mu a^\mu - \frac{2}{3}a^\mu a_\mu - \frac{2}{5}\sigma_{\mu\nu}\sigma^{\mu\nu}$$

$$\overset{1}{\mathfrak{q}}_\mu \equiv -h^\nu_\mu \frac{\mathrm{d}a_\nu}{\mathrm{d}\tau} - \frac{1}{3}D_\mu\theta + a^\nu\omega_{\mu\nu} + \frac{9}{5}a^\nu\sigma_{\mu\nu} - \frac{2}{5}D_\nu\sigma^\nu_{\ \mu}$$

$$\overset{2}{\mathfrak{q}}_{\mu\nu} \equiv h^\alpha_\mu h^\beta_\nu \frac{\mathrm{d}\sigma_{\alpha\beta}}{\mathrm{d}\tau} + D_{\langle\mu}a_{\nu\rangle} + a_{\langle\mu}a_{\nu\rangle} - 2\sigma_{\alpha(\mu}\omega^\alpha_{\ \nu)} - \frac{6}{7}\sigma_{\alpha\langle\mu}\sigma^\alpha_{\ \nu\rangle}$$

$$\overset{3}{\mathfrak{q}}_{\mu\nu\rho} \equiv -D_{\langle\mu}\sigma_{\nu\rho\rangle} - 3a_{\langle\mu}\sigma_{\nu\rho\rangle}$$

$$\overset{4}{\mathfrak{q}}_{\mu\nu\rho\kappa} \equiv 2\sigma_{\langle\mu\nu}\sigma_{\rho\kappa\rangle}$$

D_μ: spatial derivative , $\langle\rangle$: trace-free part of spatial tensor

Interpretation of effective cosmological parameters

⋆ The effective deceleration parameter

Multipole expansion of $\mathfrak{Q}$ in direction of incoming light e^μ:

$$\mathfrak{Q} = -1 - \frac{\overset{0}{\mathfrak{q}} + e^\mu \overset{1}{\mathfrak{q}}_\mu + e^\mu e^\nu \overset{2}{\mathfrak{q}}_{\mu\nu} + e^\mu e^\nu e^\rho \overset{3}{\mathfrak{q}}_{\mu\nu\rho} + e^\mu e^\nu e^\rho e^\kappa \overset{4}{\mathfrak{q}}_{\mu\nu\rho\kappa}}{\mathfrak{H}^2(\boldsymbol{e})} \,,$$

$$\overset{0}{\mathfrak{q}} \equiv \frac{1}{3}\frac{\mathrm{d}\theta}{\mathrm{d}\tau} + \frac{1}{3} D_\mu a^\mu - \frac{2}{3} a^\mu a_\mu - \frac{2}{5}\sigma_{\mu\nu}\sigma^{\mu\nu}$$

$$\overset{1}{\mathfrak{q}}_\mu \equiv -h^\nu_\mu \frac{\mathrm{d}a_\nu}{\mathrm{d}\tau} - \frac{1}{3} D_\mu \theta + a^\nu \omega_{\mu\nu} + \frac{9}{5} a^\nu \sigma_{\mu\nu} - \frac{2}{5} D_\nu \sigma^\nu_{\ \mu}$$

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$$\overset{3}{\mathfrak{q}}_{\mu\nu\rho} \equiv -D_{\langle\mu}\sigma_{\nu\rho\rangle} - 3a_{\langle\mu}\sigma_{\nu\rho\rangle}$$

$$\overset{4}{\mathfrak{q}}_{\mu\nu\rho\kappa} \equiv 2\sigma_{\langle\mu\nu}\sigma_{\rho\kappa\rangle}$$

$\frac{1}{3}\frac{\mathrm{d}\theta}{\mathrm{d}\tau}$: local acceleration of length scales;
only non-vanishing term in the comoving FLRW limit.

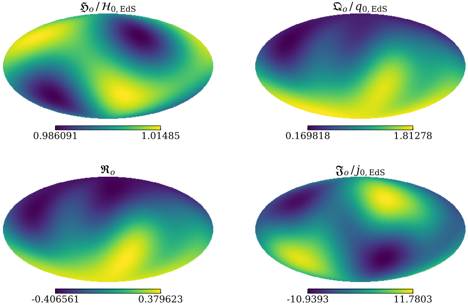
The effective deceleration parameter $\mathfrak{Q}$ does not directly measure deceleration of length scales!

Luminosity distance, **A numerical relativity study**

H. J. Macpherson and AH, Phys.Rev.D 104 (2021) 023525 [arXiv:2103.11918]
J. Adamek, C. Clarkson, R. Durrer, AH, M. Kunz, H. J. Macpherson
Open J.Astrophys. 7 (2024) [arXiv:2402.12165]

General relativistic hydrodynamical simulation (Einstein Toolkit) with cosmological initial conditions.

Sky map of effective cosmological parameters of typical observer (coarsegraining scale of ∼200Mpc/h scale corresponding to ∼ 5% density contrasts):



Cosmological drift effects

Cosmological drift effects are measurements in real time of changes to cosmic observables

Examples

Redshift drift: Change in redshift of source with proper time of the observer

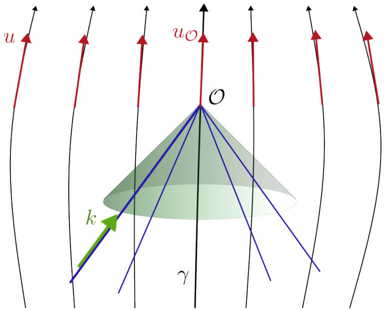
- ⋆ Lyman-α forest from quasars (ELT)
- ⋆ Neutral hydrogen 21-cm emission lines (SKA)

Position drift: Change in angular position of source with proper time of the observer

- ⋆ Proper motions measured by Gaia

Cosmological drift effects

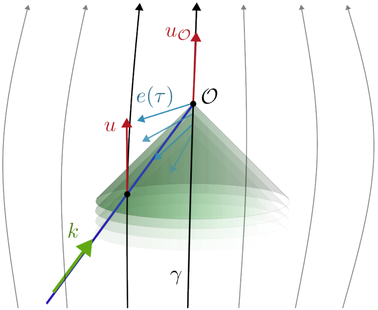
Normal way to think of observables in cosmology: One instance of our lightcone probed



AH and M. Korzyński Phys. Rev. D **110** (2024), 043525

Cosmological drift effects

In the case of cosmological drift observables: Multiple instances of our lightcone probed. Tracking the same observers over time → Non-trivial boundary value problem to solve!



AH and M. Korzyński Phys. Rev. D **110** (2024), 043525

Redshift drift cosmography

A. Heinesen, Phys. Rev. D **104** (2021), 123527 [arXiv:2107.08674]
A. Heinesen and M. Korzyński, Phys. Rev. D **110** (2024), 043525 [arXiv:2406.06167]

Redshift drift series expansion to first order in redshift:

$$\dot{z} = -\mathfrak{d}_0 \mathfrak{H}_0 z + \mathcal{O}(z^2)$$

$$(\text{FLRW: } \dot{z} = -q_o H_o z + \mathcal{O}(z^2))$$

$H \to \mathfrak{H}$

$q \to \mathfrak{d}$

Redshift drift cosmography

A. Heinesen, Phys. Rev. D **104** (2021), 123527 [arXiv:2107.08674]
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$H \to \mathfrak{H}$	1 dof $\to$ 9 dof
$q \to \mathfrak{d}$	1 dof $\to$ 12 independent dof

Redshift drift cosmography

A. Heinesen, Phys. Rev. D **104** (2021), 123527 [arXiv:2107.08674]

Redshift drift series expansion to first order in redshift:

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$$(\text{FLRW: } \dot{z} = -q_o H_o z + \mathcal{O}(z^2))$$

$$H \to \mathfrak{H} = \frac{1}{3}\theta - e^\mu a_\mu + e^\mu e^\nu \sigma_{\mu\nu}\,, \qquad 1 \text{ dof} \to 9 \text{ dof}$$

$$q \to \mathfrak{d} = \frac{-\kappa^\mu \kappa_\mu - \Sigma^o - e^\mu \Sigma^{\boldsymbol{e}}_\mu - e^\mu e^\nu \Sigma^{\boldsymbol{ee}}_{\mu\nu} - e^\mu \kappa^\nu \Sigma^{\boldsymbol{e\kappa}}_{\mu\nu}}{\mathfrak{H}^2}\,, \qquad 1 \text{ dof} \to 12 \text{ indep. dof}$$

The generalised deceleration parameter $\mathfrak{d} \neq \mathfrak{Q}$

$\kappa^\mu \equiv h^\mu_\nu u^\alpha \nabla_\alpha e^\nu$: position *drift*. Change of angular position on the sky of the source in an unrotated reference frame

Redshift drift cosmography

Redshift drift series expansion to first order in redshift:

$$\dot{z} = -\mathfrak{d}_0 \mathfrak{H}_0 z + \mathcal{O}(z^2)$$

$$\text{(FLRW: } \dot{z} = -q_o H_o z + \mathcal{O}(z^2))$$

$$q \rightarrow \mathfrak{d} = \frac{-\kappa^\mu \kappa_\mu - \Sigma^o - e^\mu \Sigma^{\boldsymbol{e}}_\mu - e^\mu e^\nu \Sigma^{\boldsymbol{ee}}_{\mu\nu} - e^\mu \kappa^\nu \Sigma^{\boldsymbol{e\kappa}}_{\mu\nu}}{\mathfrak{H}^2}, \qquad 1 \text{ dof} \rightarrow 12 \text{ indep. dof}$$

$$\Sigma^o \equiv -\frac{1}{3} u^\mu u^\nu R_{\mu\nu} + \frac{1}{3} D_\mu a^\mu + \frac{1}{3} a^\mu a_\mu \,,$$

$$\Sigma^{\boldsymbol{e}}_\mu \equiv -\frac{1}{3} \theta a_\mu - a^\nu \sigma_{\mu\nu} + 3 a^\nu \omega_{\mu\nu} - h^\nu_\mu \dot{a}_\nu \,,$$

$$\Sigma^{\boldsymbol{ee}}_{\mu\nu} \equiv a_{\langle\mu} a_{\nu\rangle} + D_{\langle\mu} a_{\nu\rangle} - u^\rho u^\sigma C_{\rho\mu\sigma\nu} - \frac{1}{2} h^\alpha_{\langle\mu} h^\beta_{\nu\rangle} R_{\alpha\beta} \,,$$

$$\Sigma^{\boldsymbol{e\kappa}}_{\mu\nu} \equiv 2(\sigma_{\mu\nu} - \omega_{\mu\nu}) \,.$$

D_μ: spatial derivative , $\langle\rangle$: trace-free part of spatial tensor

$\mathfrak{d}$ does not directly directly measure the deceleration of length scales between test particles

Luminosity distance and redshift drift generally have different effective deceleration parameters: $\mathfrak{d} \neq \mathfrak{Q}$

Redshift drift cosmography

Redshift drift series expansion to first order in redshift:

$$\dot{z} = -\mathfrak{d}_0 \mathfrak{H}_0 z + \mathcal{O}(z^2)$$

$$(\text{FLRW: } \dot{z} = -q_o H_o z + \mathcal{O}(z^2))$$

$$q \to \mathfrak{d} = \frac{-\kappa^\mu \kappa_\mu - \Sigma^o - e^\mu \Sigma^{\boldsymbol{e}}_\mu - e^\mu e^\nu \Sigma^{\boldsymbol{ee}}_{\mu\nu} - e^\mu \kappa^\nu \Sigma^{\boldsymbol{e\kappa}}_{\mu\nu}}{\mathfrak{H}^2}\,, \qquad 1 \text{ dof} \to 12 \text{ indep. dof}$$

$$\Sigma^o \equiv -\frac{1}{3} u^\mu u^\nu R_{\mu\nu} + \frac{1}{3} D_\mu a^\mu + \frac{1}{3} a^\mu a_\mu\,,$$

$$\Sigma^{\boldsymbol{e}}_\mu \equiv -\frac{1}{3}\theta a_\mu - a^\nu \sigma_{\mu\nu} + 3 a^\nu \omega_{\mu\nu} - h^\nu_\mu \dot{a}_\nu\,,$$

$$\Sigma^{\boldsymbol{ee}}_{\mu\nu} \equiv a_{\langle\mu} a_{\nu\rangle} + D_{\langle\mu} a_{\nu\rangle} - u^\rho u^\sigma C_{\rho\mu\sigma\nu} - \frac{1}{2} h^\alpha_{\langle\mu} h^\beta_{\nu\rangle} R_{\alpha\beta}\,,$$

$$\Sigma^{\boldsymbol{e\kappa}}_{\mu\nu} \equiv 2(\sigma_{\mu\nu} - \omega_{\mu\nu})\,.$$

$u^\mu u^\nu R_{\mu\nu}$: Ricci focusing term

$u^\rho u^\sigma C_{\rho\mu\sigma\nu}$: Electric part of Weyl tensor; (Analogous to Newtonian tidal tensor: gradient of gravitational acceleration)

Position drift cosmography

AH and M. Korzyński, Phys. Rev. D **110** (2024), 043525 [arXiv:2406.06167]

$$\kappa^\sigma|_o \equiv h^\mu_\nu u^\alpha \nabla_\alpha e^\nu = {}^{(0)}\kappa^\sigma|_o - \left.\frac{{}^{(1)}\kappa^\sigma}{\mathfrak{H}}\right|_o z + \mathcal{O}(z^2)\,,$$

$${}^{(0)}\kappa^\sigma = p^\sigma{}_\mu e^\alpha\, {}^{(0)}\kappa^\mu{}_\alpha$$

$${}^{(1)}\kappa^\sigma = p^\sigma{}_\mu \left[{}^{(1)}\kappa^\mu_0 + e^\alpha\, {}^{(1)}\kappa^\mu{}_\alpha + e^\alpha e^\beta\, {}^{(1)}\kappa^\mu{}_{\alpha\beta} + e^\alpha e^\beta e^\gamma\, {}^{(1)}\kappa^\mu{}_{\alpha\beta\gamma} \right]$$

$$\begin{aligned}
{}^{(0)}\kappa^\mu{}_\alpha &= \sigma^\mu{}_\alpha + \omega^\mu{}_\alpha \\
{}^{(1)}\kappa^\mu_0 &= \frac{1}{6} h^{\alpha\beta} D_\alpha (\sigma^\mu{}_\beta + \omega^\mu{}_\beta) - \frac{1}{4} R^\mu{}_\nu u^\nu \\
{}^{(1)}\kappa^\mu{}_\alpha &= -\theta(\sigma^\mu{}_\alpha + \omega^\mu{}_\alpha) - (\sigma^\beta{}_\alpha + \omega^\beta{}_\alpha)(\sigma^\mu{}_\beta + \omega^\mu{}_\beta) \\
&\quad + \frac{1}{2} R^\mu{}_\alpha + E^\mu{}_\alpha - \frac{4}{15} h^{\gamma\beta} (\sigma^\mu{}_\gamma + \omega^\mu{}_\gamma)\sigma_{\alpha\beta} \\
{}^{(1)}\kappa^\mu{}_{\alpha\beta} &= \frac{1}{2} D_{\langle\alpha} (\sigma^\mu{}_{\beta\rangle} + \omega^\mu{}_{\beta\rangle}) - \frac{1}{2} C^\mu{}_{\alpha\beta\nu} u^\nu \\
{}^{(1)}\kappa^\mu{}_{\alpha\beta\gamma} &= -(\sigma^\mu{}_{\langle\gamma} + \omega^\mu{}_{\langle\gamma})\sigma_{\alpha\beta\rangle}
\end{aligned}$$

Position drift cosmography

AH and M. Korzyński, Phys. Rev. D **110** (2024), 043525 [arXiv:2406.06167]

$$\kappa^\sigma|_o \equiv h^\mu_\nu u^\alpha \nabla_\alpha e^\nu = {}^{(0)}\kappa^\sigma|_o - \frac{{}^{(1)}\kappa^\sigma}{\mathfrak{H}}\Big|_o z + \mathcal{O}(z^2)\,,$$

$$\begin{aligned}
{}^{(0)}\kappa^\sigma &= p^\sigma{}_\mu e^\alpha\, {}^{(0)}\kappa^\mu{}_\alpha \\
{}^{(1)}\kappa^\sigma &= p^\sigma{}_\mu \Big[{}^{(1)}\kappa^\mu_0 + e^\alpha\, {}^{(1)}\kappa^\mu{}_\alpha + e^\alpha e^\beta\, {}^{(1)}\kappa^\mu{}_{\alpha\beta} + e^\alpha e^\beta e^\gamma\, {}^{(1)}\kappa^\mu{}_{\alpha\beta\gamma} \Big]
\end{aligned}$$

$$\begin{aligned}
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&\quad + \frac{1}{2} R^\mu{}_\alpha + E^\mu{}_\alpha - \frac{4}{15} h^{\gamma\beta} (\sigma^\mu{}_\gamma + \omega^\mu{}_\gamma)\sigma_{\alpha\beta} \\
{}^{(1)}\kappa^\mu{}_{\alpha\beta} &= \frac{1}{2} D_{\langle\alpha} (\sigma^\mu{}_{\beta\rangle} + \omega^\mu{}_{\beta\rangle}) - \frac{1}{2} C^\mu{}_{\alpha\beta\nu} u^\nu \\
{}^{(1)}\kappa^\mu{}_{\alpha\beta\gamma} &= -(\sigma^\mu{}_{\langle\gamma} + \omega^\mu{}_{\langle\gamma})\sigma_{\alpha\beta\rangle}
\end{aligned}$$

Discussion

Model independent cosmological analysis of distance–redshift data, position drift, and redshift drift data is possible!

Requires sufficient data, sky coverage, and control over systematic errors in data.

Now may be the time to implement anisotropic cosmography methods

- Increase in numbers of distance indicators observed
- New cosmological measurements

Things to think about when applying cosmography to data

- The scale of observation and survey geometry – sufficient data and sky coverage to constrain relevant coefficients?
- Convergence properties of the cosmographic expansion (Work in progress on convergence in anisotropic cosmography scenarios with M. Hills and H. Macpherson)
- Careful interpreting results: Anisotropic signatures are not necessarily anomalous (especially at low redshifts, significant anisotropies are expected within ΛCDM)

Discussion

Model independent cosmological analysis of distance–redshift data, position drift, and redshift drift data is possible!

Requires sufficient data, sky coverage, and control over systematic errors in data.

Now may be the time to implement anisotropic cosmography methods

- Increase in numbers of distance indicators observed
- New cosmological measurements

Things to think about when applying cosmography to data

- The scale of observation and survey geometry – sufficient data and sky coverage to constrain relevant coefficients?
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Redshift drift as a probe of the strong energy condition

A. Heinesen, Phys. Rev. D **103** (2021) L081302 [arXiv:2102.03774]

Redshift drift in a general space-time (the *exact* signal):

$$\dot{z} = (1+z)\mathfrak{H}_o - \mathfrak{H}_e + \mathcal{S}_{e\to o} = E_e \int_{\lambda_e}^{\lambda_o} d\lambda \, \Pi$$

$$(\text{FLRW: } \dot{z} = (1+z)H_o - H_e \qquad = E_e \int_{\lambda_e}^{\lambda_o} d\lambda \, \ddot{a}/a)$$

$$\Pi = \Pi^o + e^\mu \Pi^{\boldsymbol{e}}_\mu + d^\mu \Pi^{\boldsymbol{d}}_\mu + e^\mu e^\nu \Pi^{\boldsymbol{ee}}_{\mu\nu} + e^\mu d^\nu \Pi^{\boldsymbol{ed}}_{\mu\nu} + e^\mu e^\nu e^\rho \Pi^{\boldsymbol{eee}}_{\mu\nu\rho} + e^\mu e^\nu e^\rho e^\kappa \Pi^{\boldsymbol{eeee}}_{\mu\nu\rho\kappa}$$

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e^μ: Direction of incoming light as seen by the observer

$d^\mu \equiv h^\mu_\nu e^\alpha \nabla_\alpha e^\nu$: spatially projected acceleration vector of e^μ

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$$\Pi^o \equiv -\frac{1}{3}u^\mu u^\nu R_{\mu\nu} + \frac{1}{3}D_\mu a^\mu - \frac{1}{3}a^\mu a_\mu - d^\mu d_\mu - \frac{3}{5}\sigma^{\mu\nu}\sigma_{\mu\nu} - \omega^{\mu\nu}\omega_{\mu\nu}$$

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$u^\mu u^\nu R_{\mu\nu}$: Ricci focusing term;
only non-vanishing term in FLRW limit

Redshift drift as a probe of the strong energy condition

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$u^\mu u^\nu R_{\mu\nu}$: Ricci focusing term

Now assume that $a^\mu = 0$, then

$\Pi^o \leq 0$ if $u^\mu u^\nu R_{\mu\nu} \geq 0$ (Strong Energy Condition)

$\Rightarrow \dot{z} \leq 0$ if $u^\mu u^\nu R_{\mu\nu} \geq 0$ if monopole, Π^o, is dominating

Positivity of redshift drift = detection of SEC violation (Dark energy)

Luminosity distance cosmography, **general geometry**

$$\mathfrak{Q} = -1 - \frac{\overset{0}{\mathfrak{q}} + e^{\mu}\overset{1}{\mathfrak{q}}_{\mu} + e^{\mu}e^{\nu}\overset{2}{\mathfrak{q}}_{\mu\nu} + e^{\mu}e^{\nu}e^{\rho}\overset{3}{\mathfrak{q}}_{\mu\nu\rho} + e^{\mu}e^{\nu}e^{\rho}e^{\kappa}\overset{4}{\mathfrak{q}}_{\mu\nu\rho\kappa}}{\mathfrak{H}^2(\boldsymbol{e})} ,$$

$$\begin{aligned}
\overset{0}{\mathfrak{q}} &\equiv \tfrac{1}{3}\tfrac{\mathrm{d}\theta}{\mathrm{d}\tau} + \tfrac{1}{3}D_{\mu}a^{\mu} - \tfrac{2}{3}a^{\mu}a_{\mu} - \tfrac{2}{5}\sigma_{\mu\nu}\sigma^{\mu\nu} \\
\overset{1}{\mathfrak{q}}_{\mu} &\equiv -h^{\nu}_{\mu}\tfrac{\mathrm{d}a_{\nu}}{\mathrm{d}\tau} - \tfrac{1}{3}D_{\mu}\theta + a^{\nu}\omega_{\mu\nu} + \tfrac{9}{5}a^{\nu}\sigma_{\mu\nu} - \tfrac{2}{5}D_{\nu}\sigma^{\nu}_{\ \mu} \\
\overset{2}{\mathfrak{q}}_{\mu\nu} &\equiv h^{\alpha}_{\mu}h^{\beta}_{\nu}\tfrac{\mathrm{d}\sigma_{\alpha\beta}}{\mathrm{d}\tau} + D_{\langle\mu}a_{\nu\rangle} + a_{\langle\mu}a_{\nu\rangle} - 2\sigma_{\alpha(\mu}\omega^{\alpha}_{\ \nu)} - \tfrac{6}{7}\sigma_{\alpha\langle\mu}\sigma^{\alpha}_{\ \nu\rangle} \\
\overset{3}{\mathfrak{q}}_{\mu\nu\rho} &\equiv -D_{\langle\mu}\sigma_{\nu\rho\rangle} - 3a_{\langle\mu}\sigma_{\nu\rho\rangle} \\
\overset{4}{\mathfrak{q}}_{\mu\nu\rho\kappa} &\equiv 2\sigma_{\langle\mu\nu}\sigma_{\rho\kappa\rangle}
\end{aligned}$$

D_{μ}: spatial derivative , $\langle\rangle$: trace-free part of spatial tensor

Luminosity distance cosmography, **general geometry**

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$$\overset{4}{\mathfrak{q}}_{\mu\nu\rho\kappa} \equiv 2\sigma_{\langle\mu\nu}\sigma_{\rho\kappa\rangle}$$

$\frac{1}{3}\frac{\mathrm{d}\theta}{\mathrm{d}\tau}$: local acceleration of length scales;
only non-vanishing term in the comoving FLRW limit.

The effective deceleration parameter $\mathfrak{Q}$ does not directly measure deceleration of length scales!

Redshift drift cosmography

A. Heinesen, Phys. Rev. D **104** (2021), 123527 [arXiv:2107.08674]

Redshift drift series expansion to first order in redshift:

$$\dot{z} = -\mathfrak{d}_0 \mathfrak{H}_0 z + \mathcal{O}(z^2)$$

$$(\text{FLRW: } \dot{z} = -q_o H_o z + \mathcal{O}(z^2))$$

$$H \to \mathfrak{H} = \frac{1}{3}\theta - e^\mu a_\mu + e^\mu e^\nu \sigma_{\mu\nu} \,,$$ 1 dof → 9 dof

$$q \to \mathfrak{d} = \frac{-\kappa^\mu \kappa_\mu - \Sigma^o - e^\mu \Sigma^{\boldsymbol{e}}_\mu - e^\mu e^\nu \Sigma^{\boldsymbol{ee}}_{\mu\nu} - e^\mu \kappa^\nu \Sigma^{\boldsymbol{e\kappa}}_{\mu\nu}}{\mathfrak{H}^2} \,,$$ 1 dof → 12 indep. dof

The generalised deceleration parameter $\mathfrak{d} \neq \mathfrak{Q}$

$\kappa^\mu \equiv h^\mu_\nu u^\alpha \nabla_\alpha e^\nu$: position *drift*. Change of angular position on the sky of the source in an unrotated reference frame

e^μ_o is immediately known. κ^μ_o requires precise measurements.

Redshift drift cosmography

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$$\Sigma^o \equiv -\frac{1}{3} u^\mu u^\nu R_{\mu\nu} + \frac{1}{3} D_\mu a^\mu + \frac{1}{3} a^\mu a_\mu\,,$$

$$\Sigma^{\boldsymbol{e}}_\mu \equiv -\frac{1}{3}\theta a_\mu - a^\nu \sigma_{\mu\nu} + 3 a^\nu \omega_{\mu\nu} - h^\nu_\mu \dot{a}_\nu\,,$$

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$$\Sigma^{\boldsymbol{e\kappa}}_{\mu\nu} \equiv 2(\sigma_{\mu\nu} - \omega_{\mu\nu})\,.$$

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$$\Sigma^{\boldsymbol{e\kappa}}_{\mu\nu} \equiv 2(\sigma_{\mu\nu} - \omega_{\mu\nu})\,.$$

$u^\mu u^\nu R_{\mu\nu}$: Ricci focusing term

$u^\rho u^\sigma C_{\rho\mu\sigma\nu}$: Electric part of Weyl tensor, analogous to Newtonian tidal tensor